Nonlinear dynamics of spin and charge in spin-Calogero model

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The fully nonlinear dynamics of spin and charge in spin-Calogero model is studied. The latter is an integrable one-dimensional model of quantum spin-1/2 particles interacting through inverse-square interaction and exchange. Classical hydrodynamic equations of motion are written for this model in the regime where gradient corrections to the exact hydrodynamic formulation of the theory may be neglected. In this approximation variables separate in terms of dressed Fermi momenta of the model. Hydrodynamic equations reduce to a set of decoupled Riemann-Hopf (or inviscid Burgers') equations for the dressed Fermi momenta. We study the dynamics of some nonequilibrium spin-charge configurations for times smaller than the time scale of the gradient catastrophe. We find an interesting interplay between spin and charge degrees of freedom. In the limit of large coupling constant the hydrodynamics reduces to the spin hydrodynamics of the Haldane-Shastry model.

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I. INTRODUCTION

One-dimensional (1D) models of many-body systems have been a subject of intensive research since 1970s. Due to the low dimensionality, standard perturbative approaches developed in many-body theory are often inapplicable. On the other hand some techniques specific to one spacial dimension are available and allow to treat systems of interacting particles nonperturbatively. The Fermi-liquid paradigm is replaced by the Luttinger liquid theory[1] in one dimension. One of its most striking predictions is that at low energies spin and charge degrees of freedom decouple. One can say that at low energies physical electrons exist as separate spin and charge excitations. At higher energies it is expected that spin and charge recombine into the original electrons. One can see the traces of spin-charge interaction taking into account corrections to the Luttinger liquid model arising from the finite curvature of band dispersion at Fermi energy.[1] The coupling between spin and charge in one-dimensional systems was studied both perturbatively and using integrable models available in one dimension.[2]

In this paper we study the interaction between spin and charge in another integrable model—the spin-Calogero model (sCM). This model is a spin generalization[3–5] of the well-known Calogero-Sutherland model.[6]

Calogero-Sutherland-type models occupy a special place in 1D quantum physics. They are exactly solvable (integrable) but are very special even in the family of integrable models. In particular, they can be interpreted as systems of "noninteracting" particles with fractional exclusion statistics.[6–11]

The sCM model is given by the following Hamiltonian:

$$H \equiv -\frac{\hbar^2}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\hbar^2}{2}\sum_{j\neq l}\frac{\lambda(\lambda \pm \mathbf{P}_{jl})}{(x_j - x_l)^2}, \tag{1}$$

where we took the mass of particles as a unity and $\mathbf{P}_{jl}$ is the operator that exchanges the positions of particles j and l.[3] The $\pm$ sign in the exchange term corresponds to ferromagnetic and antiferromagnetic ground state, respectively, if we are studying fermions. Similarly, it corresponds to antiferromagnetic and ferromagnetic ground state, respectively, if we are considering bosonic particles. The four cases can be summarized as

$$\text{Bosons} \rightarrow \begin{cases} + & \Rightarrow \text{ Antiferromagnetic}, \\ - & \Rightarrow \text{ Ferromagnetic}, \end{cases}$$

$$\text{Fermions} \rightarrow \begin{cases} + & \Rightarrow \text{ Ferromagnetic}, \\ - & \Rightarrow \text{ Antiferromagnetic}. \end{cases}$$

The coupling parameter λ is positive and N is the total number of particles.

As it has been already noted above that the sCM is a very special model. In particular, in contrast to more generic integrable or nonintegrable models the spin and charge in sCM are not truly separated even at low energies.[4] Of course, one can still describe the low-energy excitation spectrum of sCM by two independent harmonic fluid Hamiltonians, one for the charge and the other for spin. However, it turns out that for the sCM the spin and charge velocities are the same,[4] i.e., spin and charge do not actually separate.

Here we study the spin-Calogero model in the limit of an infinite number of particles using the hydrodynamic approach. Albeit, the collective-field theory/quantum hydrodynamics of the spinless Calogero-Sutherland model has been studied in great detail,[12–16] a complete understanding of its spin generalization is still lacking although a considerable progress has been done recently in Refs. 17 and 18.

We study the nonlinear collective dynamics of sCM in the semiclassical approximation, additionally neglecting gradient corrections to the equations of motion. This limit is justified as long as we consider configurations with small gradients of density and velocity fields. The gradientless approximation is commonly employed in studying nonlinear equations[19] and allows to study the evolution for a finite time while the first nonlinear contributions are dominant. For longer times, the

solution will inevitably evolve toward configurations with large field gradients (such as shock waves) and the gradientless approximation becomes inapplicable. Nevertheless, in the initial stage of the evolution, corrections due to gradient terms in the equations of motion can be neglected (see further discussion in the Sec. V B). We derive the gradientless hydrodynamics Hamiltonian from the Bethe-Ansatz (BA) solution of the model.

The paper is organized as follows. In Sec. II we start with the simplest spinful integrable model—a system of free fermions with spin. We briefly review the Bethe-Ansatz solution for spin-Calogero model in Sec. III and deduce the hydrodynamic Hamiltonian for the sCM from this solution in Sec. IV neglecting gradient corrections. The corresponding classical equations of motion are given in Sec. V. It is shown that variables separate and the system of hydrodynamic equations is decoupled into four independent Riemann-Hopf equations for a given special linear combinations of density and velocity fields—the dressed Fermi momenta. In Sec. VI we illustrate that in the limit of strong coupling the hydrodynamics of sCM is reduced to the hydrodynamics of Haldane-Shastry lattice spin model giving the hydrodynamic formulation of the so-called *freezing trick*.[20] We present some particular solutions of the hydrodynamic equations demonstrating nonlinear coupling between spin and charge degrees of freedom in the sCM in Sec. VII and conclude in Sec. VIII. To avoid interruptions in the main part of the paper some important technical details are moved to the appendices and are organized as follows. In Appendix A we use asymptotic Bethe ansatz to derive the hydrodynamics of sCM and to explain why variables separate in this system. In Appendix B we describe the notion of true hydrodynamic velocities. In Appendix C we relate the hydrodynamics of sCM to two infinite families of mutually commuting conserved quantities and collect our results for the hydrodynamics in the different regimes of sCM. Finally, in Appendix D we derive a hydrodynamic description of the Haldane-Shastry model (HSM) from its Bethe-Ansatz solution.

II. FREE FERMIONS WITH SPIN

For one-dimensional free fermions without internal degrees of freedom the lowest state with a given total number of particles and total momentum corresponds to all single-particle plane-wave states filled if the corresponding momentum k satisfies $k_L<k<k_R$. Here $k_{L,R}$ are left and right Fermi momenta, respectively, which are defined by the given number of particles and momentum of the system,

$$N/L=\int_{k_L}^{k_R}\frac{dk}{2\pi}=\frac{k_R-k_L}{2\pi}=\rho, \tag{2}$$

$$P/L=\int_{k_L}^{k_R}\frac{dk}{2\pi}\hbar k=\hbar\frac{k_R^2-k_L^2}{4\pi}=\rho v. \tag{3}$$

Here we introduced the (overall) velocity of the system v which is given from Eqs. (2) and (3) by

$$v/\hbar=\frac{k_R+k_L}{2}. \tag{4}$$

Inverting Eqs. (2) and (4) we express the left and right Fermi points $k_{L,R}$ in terms of the density ρ and velocity v as

$$k_{R,L}=v/\hbar\pm\pi\rho. \tag{5}$$

The energy of this state is given by

$$E/L=\int_{k_L}^{k_R}\frac{dk}{2\pi}\frac{\hbar^2k^2}{2}=\hbar^2\frac{k_R^3-k_L^3}{12\pi}=\frac{\rho v^2}{2}+\frac{\hbar^2\pi^2}{6}\rho^3. \tag{6}$$

Up to this moment $\rho,v,k_{R,L}$ are just numbers characterizing the chosen state of free fermions (only two of them are independent). Assuming the locality of the theory we promote these numbers to quantum fields and write the hydrodynamic Hamiltonian of free spinless fermions as

$$\mathcal{H}=\int dx\left[\frac{\rho(x)v^2(x)}{2}+\frac{\hbar^2\pi^2}{6}\rho^3(x)\right]$$
$$=\int dx\hbar^2\frac{[k_R(x)]^3-[k_L(x)]^3}{12\pi}. \tag{7}$$

Here we consider $\rho(x)$ and $v(x)$ as quantum field operators of density and velocity [and $k_{R,L}$ as given by Eq. (5)] having canonical commutation relations[21]

$$[\rho(x),v(y)]=-\mathrm{i}\hbar\delta'(x-y). \tag{8}$$

Of course, gradient corrections to Eq. (7) are generically present and the above "derivation" is just a heuristic argument (semiclassical in nature). It turns out that Eq. (7) is, in fact, exact for free fermions.[22] It can be derived rigorously either using the method of collective-field theory[23–25] or conventional bosonization technique (but without linearization at Fermi points).[1,26,27]

The two terms of Eq. (7) have a very clear physical interpretation. The first term is the kinetic energy of a fluid moving as a whole—the only velocity term allowed by Galilean invariance. The second one is the kinetic energy of the internal motion of particles. This term is finite due to the Pauli exclusion principle. Within the hydrodynamic approach we have to think of this term as of an internal energy of the fluid.

Commuting Hamiltonian (7) with the density and velocity operators one obtains the continuity and the Euler equations of quantum hydrodynamics. Alternatively, using $[k_L(x),k_L(y)]=-[k_R(x),k_R(y)]=2\pi i\delta'(x-y)$ the equations of motion can also be written as a system of quantum Riemann-Hopf equations

$$\dot{k}_{R,L}+\hbar k_{R,L}\partial_x k_{R,L}=0. \tag{9}$$

For free fermions with spin, we simply add Hamiltonians (7) written for spin-up and spin-down fermions,

$$H=\int dx\left\{\frac{1}{2}\rho_\uparrow v_\uparrow^2+\frac{1}{2}\rho_\downarrow v_\downarrow^2+\frac{\pi^2\hbar^2}{6}(\rho_\uparrow^3+\rho_\downarrow^3)\right\}. \tag{10}$$

Expanding Eq. (10) around the background density $\rho_0=\frac{k_F}{\pi}$ and the background velocity $v_0=0$ up to quadratic terms in v_α and $\delta\rho_\alpha=\rho_\alpha-\rho_0$, we obtain the harmonic fluid approximation

$$H \approx \frac{\rho_0}{2}\int dx(v_\uparrow^2+\pi^2\hbar^2\delta\rho_\uparrow^2+v_\downarrow^2+\pi^2\hbar^2\delta\rho_\downarrow^2)$$

$$\approx \frac{\rho_0}{4}\hbar^2\sum_{\alpha=\uparrow,\downarrow}\int dx[(\partial_x\phi_{R,\alpha})^2+(\partial_x\phi_{L,\alpha})^2] \tag{11}$$

with right and left bosonic fields defined as $\partial_x\phi_{R(L),\alpha}=v_\alpha/\hbar\pm\pi\delta\rho_\alpha$. This procedure is equivalent to the conventional linear bosonization procedure where the fermionic spectrum is linearized at the Fermi points.

In the spin-charge basis,

$$\rho_{c,s}\equiv\rho_\uparrow\pm\rho_\downarrow \quad \text{and} \quad v_{c,s}=\frac{v_\uparrow\pm v_\downarrow}{2}, \tag{12}$$

the harmonic theory [Eq. (11)] is described by a sum of two independent harmonic fluid Hamiltonians, one for charge and the other for spin degrees of freedom

$$H \approx \frac{\rho_0}{4}\int dx(4v_c^2+\pi^2\hbar^2\delta\rho_c^2+4v_s^2+\pi^2\hbar^2\delta\rho_s^2). \tag{13}$$

After linearization, the quantum Riemann-Hopf Eq. (9) reduces to (where $\pm$ stands for $\chi=\{R,L\}$ respectively)

$$\dot{k}_{\alpha,\chi}\pm\hbar\pi\rho_0\partial_x k_{\alpha,\chi}=0, \quad \alpha=\{\uparrow,\downarrow\}; \tag{14}$$

from which we identify that the quadratic excitations propagate like wave equations with sound velocities $u_{\text{charge}}=u_{\text{spin}}=\pi\hbar\rho_0$, equal for spin and charge. Turning on interactions between fermions generally renormalizes spin and charge sound velocities differently and results in genuine spin-charge separation at the level of harmonic approximation. The spin-Calogero-Sutherland model happens to be very special in this respect. Despite a nontrivial interaction for spin and charge, their sound velocities remain the same.

Although spin and charge are not truly separated for a free fermion system (and for the sCM), the interaction between spin and charge is absent at the level of harmonic approximation (13). This interaction appears if nonlinear corrections to Eq. (13) are taken into account [e.g., by the fully nonlinear Hamiltonian (10)] and due to gradient corrections to the hydrodynamics. The latter are not considered in this paper.

In the proper classical limit $\hbar\to 0$ all terms of Eq. (10) but the velocity terms vanish (Fermi statistics does not exist for classical particles). Instead, we are interested in a "semiclassical" limit in which $\rho\sim v/\hbar$. In this limit we rescale time and velocity by $\hbar$ ($t\to t/\hbar$ and $v\to\hbar v$) and measure everything in length units. This is equivalent to dropping all $\hbar$ from equations. For instance, Hamiltonian (10) becomes

$$H=\int dx\left\{\frac{1}{2}\rho_\uparrow v_\uparrow^2+\frac{1}{2}\rho_\downarrow v_\downarrow^2+\frac{\pi^2}{6}(\rho_\uparrow^3+\rho_\downarrow^3)\right\}. \tag{15}$$

We replace the commutation relations [Eq. (8)] by the corresponding classical Poisson brackets (for up and down species)

$$\{\rho_\alpha(x),v_\beta(y)\}=\delta_{\alpha\beta}\delta'(x-y) \tag{16}$$

and consider the classical equations of motion generated by the Hamiltonian together with the Poisson brackets. In the remainder of the paper all hydrodynamic equations are obtained in this *semiclassical* limit.

III. SPIN-CALOGERO MODEL

In this work we concentrate on the hydrodynamics of sCM [Eq. (1)] for the case of spin-1/2 fermions with an antiferromagnetic sign of interaction. It is convenient to impose periodic boundary conditions, i.e., consider particles living on a ring of the length L. This Hamiltonian is given by

$$H=-\frac{\hbar^2}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}+\frac{\hbar^2}{2}\left(\frac{\pi}{L}\right)^2\sum_{j\neq l}\frac{\lambda(\lambda-\mathbf{P}_{jl})}{\sin^2\frac{\pi}{L}(x_j-x_l)} \tag{17}$$

and is known to be integrable.[6] All eigenstates of Eq. (17) can be enumerated by the distribution function

$$\nu(\kappa)=\nu_\uparrow(\kappa)+\nu_\downarrow(\kappa). \tag{18}$$

Here, κ are integer-valued quantum numbers identifying a given state in a Bethe-Ansatz description and $\nu_{\uparrow,\downarrow}(\kappa)=0,1$ depending on whether a given κ is present in the solution of the Bethe-Ansatz equations.

The total momentum P and energy E of the eigenstate are given in terms of the distribution function $\nu(\kappa)$ as[28,29]

$$P=\left(\frac{2\pi}{L}\right)\sum_{\kappa=-\infty}^{+\infty}\kappa\nu(\kappa), \tag{19}$$

$$E=E_0+\left(\frac{1}{2}\right)\left(\frac{2\pi}{L}\right)^2\epsilon, \tag{20}$$

$$\epsilon=\sum_{\kappa=-\infty}^{+\infty}\kappa^2\nu(\kappa)+\frac{\lambda}{2}\sum_{\kappa,\kappa'}|\kappa-\kappa'|\nu(\kappa)\nu(\kappa'), \tag{21}$$

where $E_0=\frac{\pi^2\lambda^2}{6}N(N^2-1)$ is the energy of a reference state.[29] The numbers of particles with spin up and spin down are separately conserved in Eq. (17) and are given by

$$N_{\uparrow,\downarrow}=\sum_{\kappa=-\infty}^{+\infty}\nu_{\uparrow,\downarrow}(\kappa). \tag{22}$$

The ground-state wave function for Eq. (17) is[4,29]

$$\psi_{\text{GS}}=\prod_{j<l}\left|\sin\frac{\pi}{L}(x_j-x_l)\right|^\lambda\prod_{j<l}\left[\sin\frac{\pi}{L}(x_j-x_l)\right]^{\delta(\sigma_j,\sigma_l)}$$
$$\times\exp\left[i\frac{\pi}{2}\text{sgn}(\sigma_j-\sigma_l)\right] \tag{23}$$

and corresponds to the distributions[30]

$$\nu_\uparrow(\kappa)=\theta(-N_\uparrow/2<\kappa<N_\uparrow/2),$$

$$\nu_\downarrow(\kappa)=\theta(-N_\downarrow/2<\kappa<N_\downarrow/2). \tag{24}$$

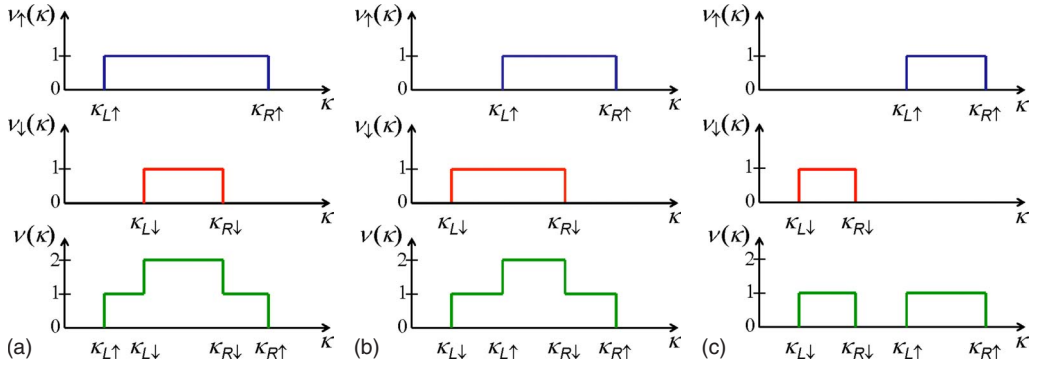

FIG. 1. (Color online) Distribution functions are shown for the three nonequivalent regimes: *complete overlap* in (a), *partial overlap* in (b), and *no overlap* in (c). Three additional regimes exist but are physically equivalent to the ones considered in these pictures and can be obtained by exchanging $\uparrow \leftrightarrow \downarrow$.

IV. GRADIENTLESS HYDRODYNAMICS OF SPIN-CALOGERO MODEL

Following the example of free fermions, we consider a uniform state specified by the following distributions:

$$\nu_\uparrow(\kappa) = \theta(\kappa_{L\uparrow} < \kappa < \kappa_{R\uparrow}), \tag{25}$$

$$\nu_\downarrow(\kappa) = \theta(\kappa_{L\downarrow} < \kappa < \kappa_{R\downarrow}). \tag{26}$$

This state is the lowest-energy state with given numbers of particles, momentum, and total spin current. It is specified by four integer numbers $\kappa_{L,R;\uparrow,\downarrow}$. All physical quantities such as energy, momentum, and higher integrals of motion of the state can be expressed in terms of these numbers using Eqs. (22), (19), and (20). These conserved quantities written as integrals over constant quantities are

$$N_\alpha = \int dx \rho_\alpha = \frac{2\pi}{L} \int dx \left[\frac{\kappa_{R\alpha} - \kappa_{L\alpha}}{2\pi} \right], \quad \alpha = \{\uparrow, \downarrow\}, \tag{27}$$

$$P = \int dx j_c = \left(\frac{2\pi}{L} \right)^2 \sum_{\alpha=\{\uparrow,\downarrow\}} \int dx \left[\frac{\kappa_{R\alpha}^2 - \kappa_{L\alpha}^2}{4\pi} \right]. \tag{28}$$

Comparison with Eqs. (2) and (3) suggests the following hydrodynamic identifications:

$$v_\uparrow \pm \pi\rho_\uparrow \equiv \frac{2\pi}{L} \kappa_{(R,L);\uparrow}, \tag{29}$$

$$v_\downarrow \pm \pi\rho_\downarrow \equiv \frac{2\pi}{L} \kappa_{(R,L);\downarrow}. \tag{30}$$

In the main body of the paper we use $v_{\uparrow,\downarrow}$ and refer to them as to "velocities." At this point they have been introduced "by analogy" with the case of free fermions. In Appendices A–C we show that these velocities are indeed conjugated to the corresponding densities and explain their relations to the true hydrodynamic velocities. In fact, in the most interesting case to us, namely, the complete overlap (CO) regime (see below) these velocities coincide with the true hydrodynamic velocities defined in Appendix B. The total momentum [Eq. (28)] of the system in terms of Eqs. (29) and (30) is

$$P = \int dx (\rho_\uparrow v_\uparrow + \rho_\downarrow v_\downarrow). \tag{31}$$

One can also express the energy [Eq. (20)] in terms of these hydrodynamic variables. Because of the nonanalyticity (presence of an absolute value) in formula (20) it is convenient to consider different physical regimes. These regimes are defined by the mutual arrangement of the supports of the distribution functions [Eqs. (25) and (26)]. There are six different regimes that reduce to three physically nonequivalent ones using the permutation $\uparrow \leftrightarrow \downarrow$. The distributions corresponding to different regimes are shown in Fig. 1:

(1) *Complete Overlap (CO) regime*. The support of $\nu_\downarrow$ is completely contained in $\nu_\uparrow$ (or vice versa). This is the regime considered in Ref. 4, where its exact solution was given.

(2) *Partial Overlap (PO) regime*. The supports of $\nu_\uparrow$ and of $\nu_\downarrow$ only partially overlap.

(3) *No Overlap (NO) regime*. The supports of $\nu_\uparrow$ and of $\nu_\downarrow$ do not overlap at all.

Notice that the small fluctuations around the singlet ground state (with $\rho_s = 0$) belong to first two regimes.

In terms of the hydrodynamic variables the three regimes are summarized in Fig. 2 and are defined by the following inequalities:

$$\text{Complete overlap} \rightarrow |v_s| < \frac{\pi}{2} |\rho_s|, \tag{32}$$

$$\text{Partial overlap} \rightarrow \frac{\pi}{2} |\rho_s| < |v_s| < \frac{\pi}{2} \rho_c, \tag{33}$$

$$\text{No overlap} \rightarrow \frac{\pi}{2} \rho_c < |v_s|, \tag{34}$$

where we switched to the spin and charge degrees of freedom defined by Eq. (12).

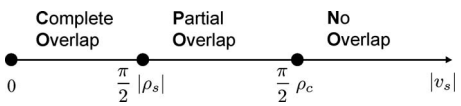

FIG. 2. Diagram capturing all cases.

To simplify the presentation we give here formulas only for the CO regime

$$-\frac{\pi\rho_s}{2} < v_s < \frac{\pi\rho_s}{2}, \tag{35}$$

where we also assumed that $\rho_s > 0$. The opposite case $\rho_s < 0$ can be obtained exchanging up and down variables. The other regimes and formulas valid for all regimes are considered in detail in Appendix C.

In the CO regime [Eq. (35)], the Hamiltonian can be written as

$$H_{\mathrm{CO}} = \int dx \left\{ \frac{1}{2}\rho_\uparrow v_\uparrow^2 + \frac{1}{2}\rho_\downarrow v_\downarrow^2 + \frac{\lambda}{2}\rho_\downarrow (v_\uparrow - v_\downarrow)^2 + \frac{\pi^2\lambda^2}{6}\rho_c^3 + \frac{\pi^2}{6}(\rho_\uparrow^3 + \rho_\downarrow^3) + \frac{\lambda\pi^2}{6}(2\rho_\uparrow^3 + 3\rho_\uparrow^2\rho_\downarrow + 3\rho_\downarrow^3) \right\}. \tag{36}$$

It is obtained by expressing Eqs. (20) and (21) in terms of hydrodynamic variables [Eqs. (29) and (30)] using Eq. (35). As in the case of free fermions (see Sec. II) we now consider $\rho_{\uparrow,\downarrow}(x,t)$ and $v_{\uparrow,\downarrow}(x,t)$ as space and time-dependent classical hydrodynamic fields with Poisson brackets [Eq. (16)]. Of course, going from the energy of the uniform state [Eqs. (25) and (26)] to the nonuniform hydrodynamic state we neglected gradients of density and velocity fields. We refer to this approximation as to *gradientless hydrodynamics*. The equations of motion generated by Hamiltonian (36) with Poisson brackets [Eq. (16)] can be used only when gradients can be neglected as compared to the gradientless terms. This means that one can use this gradientless hydrodynamics only at relatively small times (compared to the time of the gradient catastrophe, see the discussion below).

Before analyzing more general case let us consider some special limits of Eq. (36).

A. Spinless limit

In the fully polarized state $\rho_\downarrow = 0$ we obtain from Eq. (36) the gradientless Hamiltonian for spinless Calogero-Sutherland model

$$H^{\mathrm{spinless}} = \int_{-\infty}^{+\infty} dx \left\{ \frac{1}{2}\rho v^2 + \frac{\pi^2}{6}(\lambda+1)^2\rho^3 \right\}, \tag{37}$$

where we dropped the subscript $\uparrow$. The hydrodynamics [Eq. (37)] was used in Ref. 31 to calculate the leading term of an asymptotics of a particular correlation function (emptiness formation probability) for the Calogero-Sutherland model. It can be, of course, obtained by dropping gradient terms in the exact hydrodynamics derived using collective-field theory.[13–15]

B. $\lambda = 0$—free fermions with spin

At the particular value $\lambda = 0$ the sCM reduces to free fermions with spin and Hamiltonian (36) becomes the collective Hamiltonian for free fermions [Eq. (15)].

C. $\lambda \to \infty$ limit

In the limit of large coupling constant $\lambda \to \infty$ the particles form a rigid-lattice and charge degrees of freedom essentially get frozen.[20] We expect to arrive to an effective spin dynamics equivalent to the Haldane-Shastry model[32,33] (see Appendix D). This reduction to the Haldane-Shastry model is usually referred to as *freezing trick*.[20] We analyze this reduction in more detail in Sec. VI.

V. EQUATIONS OF MOTION AND SEPARATION OF VARIABLES

A. Equations of motion

The classical gradientless hydrodynamics for sCM is given by Hamiltonian (36) with canonic Poisson's brackets [Eq. (16)]. The classical evolution equations generated by this Hamiltonian are

$$\dot{\rho}_\uparrow = -\partial_x\{\rho_\uparrow v_\uparrow + \lambda\rho_\downarrow(v_\uparrow - v_\downarrow)\},$$

$$\dot{\rho}_\downarrow = -\partial_x\{\rho_\downarrow v_\downarrow - \lambda\rho_\downarrow(v_\uparrow - v_\downarrow)\},$$

$$\dot{v}_\uparrow = -\partial_x\left\{ \frac{v_\uparrow^2}{2} + \frac{\pi^2\lambda^2}{2}(\rho_\uparrow + \rho_\downarrow)^2 + \lambda\pi^2(\rho_\uparrow^2 + \rho_\uparrow\rho_\downarrow) + \frac{\pi^2}{2}\rho_\uparrow^2 \right\},$$

$$\dot{v}_\downarrow = -\partial_x\left\{ \frac{v_\downarrow^2}{2} + \frac{\lambda}{2}(v_\uparrow - v_\downarrow)^2 + \frac{\pi^2\lambda^2}{2}(\rho_\uparrow + \rho_\downarrow)^2 + \frac{\lambda\pi^2}{2}(\rho_\uparrow^2 + 3\rho_\downarrow^2) + \frac{\pi^2}{2}\rho_\downarrow^2 \right\}. \tag{38}$$

This is the system of continuity and Euler's equations for two coupled fluids (with spin up and spin down). We can also rewrite it in terms of spin and charge variables [Eq. (12)]

$$\dot{\rho}_c = -\partial_x\{\rho_c v_c + \rho_s v_s\},$$

$$\dot{\rho}_s = -\partial_x\{\rho_s(v_c - 2\lambda v_s) + (2\lambda+1)\rho_c v_s\},$$

$$\dot{v}_c = -\partial_x\left\{ \frac{v_c^2}{2} + (2\lambda+1)\frac{v_s^2}{2} + \frac{\pi^2}{8}[(2\lambda+1)^2\rho_c^2 + (2\lambda+1)\rho_s^2] \right\},$$

$$\dot{v}_s = -\partial_x\left\{ v_c v_s - \lambda v_s^2 + \frac{\pi^2}{4}\rho_s[(2\lambda+1)\rho_c - \lambda\rho_s] \right\}. \tag{39}$$

One can see that spin and charge are not decoupled. It turns out, however, that the variables nevertheless separate and the system of four coupled Eq. (38) can be written as four decoupled Riemann-Hopf equations [similar to Eq. (9)] for a special linear combinations of density and velocity fields. In the following we study the interaction of spin and charge governed by the above equations.

B. Free fermions ($\lambda = 0$) and Riemann-Hopf equation

At $\lambda = 0$ Eq. (38) becomes the hydrodynamic equations for free fermions. Fluids corresponding to up and down spin are completely decoupled

$$\dot{\rho}_{\uparrow,\downarrow} = -\partial_x\{\rho_{\uparrow,\downarrow} v_{\uparrow,\downarrow}\}, \tag{40}$$

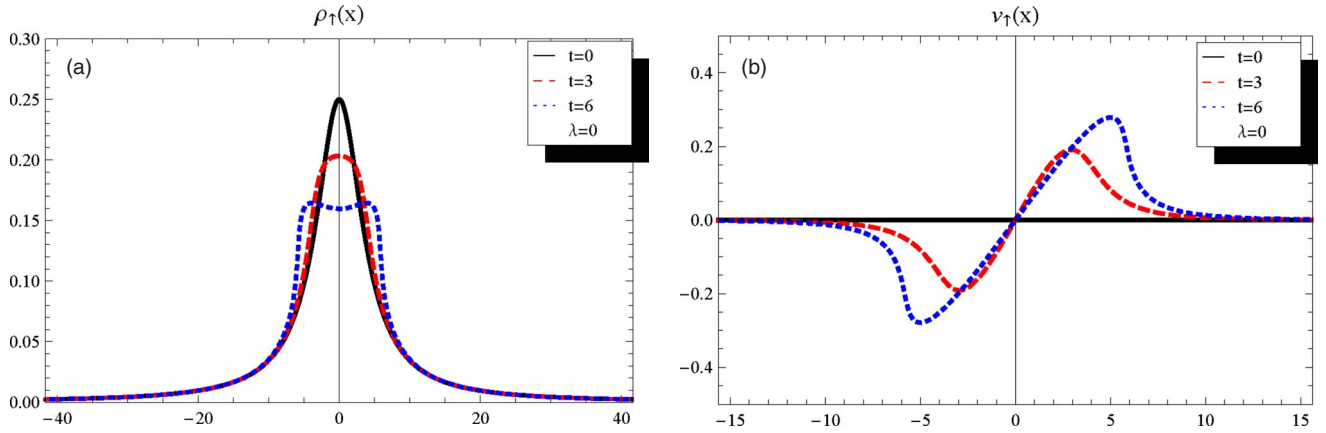

FIG. 3. (Color online) Dynamics of density field $\rho_\uparrow(x)$ (left panel) and of velocity field $v_\uparrow(x)$ (right panel) for free a fermion case ($\lambda=0$). The initial density profile at $t=0$ is a Lorentzian (46) of height $h=0.25$ and half width $a=4$. The initial velocity is zero.

$$\dot{v}_{\uparrow,\downarrow} = -\partial_x \left\{ \frac{1}{2} v_{\uparrow,\downarrow}^2 + \frac{\pi^2}{2} \rho_{\uparrow,\downarrow}^2 \right\}. \tag{41}$$

Let us introduce the following linear combinations of densities and velocities

$$k_{R\uparrow,L\uparrow} = v_\uparrow \pm \pi \rho_\uparrow,$$

$$k_{R\downarrow,L\downarrow} = v_\downarrow \pm \pi \rho_\downarrow. \tag{42}$$

These combinations are nothing else but right and left Fermi momenta of free fermions. All of them satisfy the so-called Riemann-Hopf equation

$$u_t + u u_x = 0. \tag{43}$$

The equation is the same for all four combinations $u=k_{R,L;\uparrow,\downarrow}$ and the system [Eqs. (40) and (41)] is equivalent to four decoupled Riemann-Hopf equations.

The Riemann-Hopf Eq. (43) is easily solvable with the general solution given implicitly by

$$u = u_0(x - ut). \tag{44}$$

Here $u_0(x)$ is an initial profile of $u(x,t)$ at $t=0$. One should solve Eq. (44) with respect to u and find $u(x,t)$—the solution of Eq. (43) with $u(x,t=0)=u_0(x)$. The solution (44) can also be written in a parametric form

$$x = y + t u_0(y),$$

$$u(x,t) = u_0(y). \tag{45}$$

This solution corresponds to the "Lagrangian picture" of fluid dynamics and states that points in the $x-u$ plane are just translated along x with velocity u, i.e., $(x,u_0) \to (x+tu_0,u_0)$. This picture is especially useful to solve Eq. (43) numerically.

We notice here that the nonlinear dynamics [Eq. (43)] without dispersive (higher gradient) terms is ill defined at large times. For any initial profile $u_0(x)$, at large times $t>t_c$ infinite gradients u_x will develop—*gradient catastrophe*—and solutions of Eq. (44) will become multivalued. The classical Eq. (43) will not have a meaning for $t>t_c$. We refer to the time t_c (function of the initial profile) as to the gradient catastrophe time. The gradientless hydrodynamics is applicable only for times smaller that t_c.[34] We will discuss in more detail about validity of gradientless hydrodynamics in Sec. VII.

We present a simple illustration of the density and velocity dynamics for free fermion system in Fig. 3. It is enough to consider only up spin as the evolution of up and down spins is decoupled. We chose the initial profile of the density as Lorentzian with the half width a and height h

$$\rho_{0\uparrow}(x) = \frac{h}{1 + (x/a)^2} \tag{46}$$

and an initial velocity zero. We find the initial profiles of $k_{\uparrow;R,L}$ using Eq. (42). Then we solve the Riemann-Hopf Eq. (43) using Eq. (45) and find the density and velocity at any time inverting Eq. (42). We remark that for an arbitrary smooth bump of height h and width a the gradient catastrophe time can be estimated as $t_c \approx \frac{a}{h}$. For the evolution given by Eq. (43) with an initial Lorentzian profile [$u_0(x)$ given by Eq. (46)] one can compute the gradient catastrophe time exactly. An infinite gradient $\partial_x u \to \infty$ develops at the time

$$t_c = \frac{8}{3\sqrt{3}} \frac{a}{h}. \tag{47}$$

For arbitrary initial conditions we compute the gradient catastrophe time numerically.

C. Riemann-Hopf equations for sCM

Although the system of Eq. (38) is a system of four coupled nonlinear equations, it allows for a separation of variables. Introducing the linear combinations of fields

$$k_{R\uparrow,L\uparrow} = v_\uparrow \pm \pi[(\lambda+1)\rho_\uparrow + \lambda \rho_\downarrow] = (v_\uparrow \pm \pi \rho_\uparrow) \pm \lambda \pi \rho_c, \tag{48}$$

$$\begin{aligned} k_{R\downarrow,L\downarrow} &= (\lambda+1) v_\downarrow - \lambda v_\uparrow \pm \pi(2\lambda+1)\rho_\downarrow \\ &= (v_\downarrow \pm \pi \rho_\downarrow) + \lambda(-2v_s \pm 2\pi\rho_\downarrow). \end{aligned} \tag{49}$$

we obtain the Riemann-Hopf Eq. (43) separately for all four $u=k_{L,R,\uparrow,\downarrow}$.

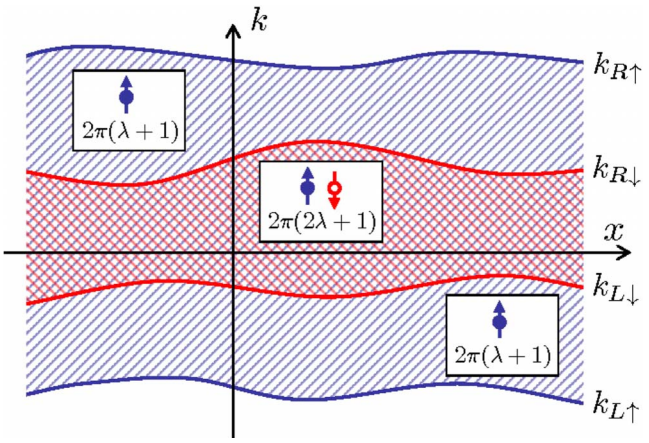

FIG. 4. (Color online) Phase-space diagram of a hydrodynamic state characterized by four space-dependent Fermi momenta.

This property of variable separation is shared with the free fermion case Sec. V B. We notice, however, that in the case of sCM, variables separate only in gradientless approximation. The gradient terms neglected in this paper will couple the hydrodynamic equations in an essential nonseparable[35] way.

The separation of variables in terms of Eqs. (48) and (49) is not so surprising. One can recognize Eqs. (48) and (49) as dressed (physical) "Fermi" momenta of (asymptotic) Bethe Ansatz. The integrals of motion of sCM are separated in terms of these Fermi momenta and the same is true for the equations of motion. We do not interrupt the presentation with this connection with the Bethe-Ansatz solution of sCM but devote the Appendix A to this purpose.

It is convenient to summarize the gradientless hydrodynamics of sCM by the picture in a "single-particle" phase space showing space-dependent Fermi momenta.[36] We plot the space-dependent Fermi momenta in an $x-k$ plane as four smooth lines. In the CO regime considered here (see Appendix C for other regimes) the Fermi momenta are ordered as

$$k_{L\uparrow}(x) < k_{L\downarrow}(x) < k_{R\downarrow}(x) < k_{R\uparrow}(x). \tag{50}$$

We fill the space between those lines with particles obeying the following rules of particles with fractional exclusion statistics[11] (see Appendix A) (i) each particle occupies a phase-space volume $2\pi(\lambda+1)$ if there are no particles of the other species in this volume, (ii) two particles with opposite spins occupy a phase-space volume $2\pi(2\lambda+1)$ [or $2\pi(\lambda+1/2)$ per particle]. The velocity $v_\uparrow(x)$ is visualized as a center of a spin-up stripe on Fig. 4 [see Eq. (B5)]. The interpretation of $v_\downarrow(x)$ is a bit less straightforward. It should be thought as a weighted average of positions of centers of both stripes [Eq. (B5)].

VI. FREEZING TRICK AND HYDRODYNAMICS OF HALDANE-SHASTRY MODEL

Here we consider the limit of large coupling constant $\lambda\to\infty$. In this limit we expect that particles form a one-dimensional lattice and only spin dynamics is important at low energies. We refer to this limit as to a freezing of the charge. We are interested in fluctuations around the uniform state with a given charge density. It can be seen from Fig. 4 that particles occupy the volume $2\pi(\lambda+1/2)$ of the phase space when both species are present. Therefore, the natural expansion parameter is $\mu=\lambda+1/2$ instead of λ.[37] We will see that the leading in μ term of the dynamics results in charge freezing while the next to leading term gives the nontrivial spin dynamics of the lattice model known as the Haldane-Shastry model[32,33]

$$H_{\mathrm{HSM}} = 2\sum_{j<l}\frac{S_j\cdot S_l}{(j-l)^2}. \tag{51}$$

This model is known to be integrable.[6] The freezing procedure described is referred to as "freezing trick" and was introduced by Polychronakos.[20] Our goal is to implement the procedure in a hydrodynamic description.

Before proceeding to a regular expansion of the equations of motion we start with a heuristic argument. We rewrite the hydrodynamic Hamiltonian (36) in terms of spin and charge variables [Eq. (12)] and consider first the two leading terms in a $1/\mu$ expansion

$$H = \int dx\left\{\frac{1}{2}\rho_c v_c^2 + \rho_s v_c v_s + \mu\rho_c v_s^2 - \left(\mu-\frac{1}{2}\right)\rho_s v_s^2 + \frac{\pi^2\mu^2}{6}\rho_c^3 + \frac{\pi^2}{4}\mu\rho_c\rho_s^2 - \frac{\pi^2}{12}\left(\mu-\frac{1}{2}\right)\rho_s^3\right\}s \tag{52}$$

$$= \int dx\left\{\frac{\pi^2}{6}\mu^2\rho_c^3 + \mu\left[\rho_c v_s^2 - \rho_s v_s^2 + \frac{\pi^2\rho_c\rho_s^2}{4} - \frac{\pi^2\rho_s^3}{12}\right] + O(1)\right\}. \tag{53}$$

The first term proportional to μ^2 comes from the energy of a static lattice while the second term proportional to μ gives the Hamiltonian of the Haldane-Shastry model in the hydrodynamic formulation (see Appendix D), i.e., describes the spin dynamics. Note that ρ_c here should be considered as a constant equal to the inverse lattice spacing of the charge lattice.

To build a systematic expansion in $1/\mu$ we go to the hydrodynamic evolution equations given in Eq. (39). We introduce the following series in $1/\mu=1/(\lambda+1/2)$ for the space-time-dependent fields.

$$u = u^{(0)} + \frac{1}{\mu}u^{(1)} + \frac{1}{\mu^2}u^{(2)} + \cdots$$

$$u \to \rho_c, v_c, \rho_s, v_s$$

and rescale time $t=\tau/\mu$ (or $\partial_t=\mu\partial_\tau$). We substitute these expansions into Eq. (39) and compare order by order in μ. Let us consider few leading orders explicitly.

A. $O(\mu)$

In this order the only nontrivial equation gives

$$0 = -\partial_x[\rho_c^{(0)2}] \tag{54}$$

and implies that $\rho_c^{(0)}$ is constant in space.

B. $O(1)$

At this order we have

$$\dot{\rho}_c^{(0)} = 0, \tag{55}$$

$$\dot{\rho}_s^{(0)} = -\partial_x\{2\rho_c^{(0)}v_s^{(0)} - 2\rho_s^{(0)}v_s^{(0)}\}, \tag{56}$$

$$\dot{v}_c^{(0)} = -\partial_x\left\{v_s^{(0)2} + \pi^2\rho_c^{(0)}\rho_c^{(1)} + \frac{\pi^2}{4}\rho_s^{(0)2}\right\}, \tag{57}$$

$$\dot{v}_s^{(0)} = -\partial_x\left\{-v_s^{(0)2} + \frac{\pi^2}{2}\rho_c^{(0)}\rho_s^{(0)} - \frac{\pi^2}{4}\rho_s^{(0)2}\right\}. \tag{58}$$

Combining Eqs. (54) and (55) we see that $\rho_c^{(0)}$ is a constant independent of space time. The evolution Eq. (56) for spin density, $\dot{\rho}_s^{(0)}$ and Eq. (58) for the spin velocity $\dot{v}_s^{(0)}$ do not depend on the dynamics of the charge and are precisely the ones obtained for the Haldane-Shastry model [compare to Eq. (D17)]. We refer the reader to the Appendix D for more details on the hydrodynamics of the Haldane-Shastry model.

Equation (57) is important in resolving a well-known "paradox." In the original spin-Calogero model the momentum of the system is identical to the total charge current since all particles in the model have the same charge. On the other hand in the Haldane-Shastry model the momentum is carried by spin excitations and superficially no charge motion is involved. One can ask how this is compatible with getting the Haldane-Shastry model in the limit $\lambda \to \infty$ from the spin-Calogero model. Equation (57) is necessary to make sure that the the current density $j(x)=\rho_c v_c + \rho_s v_s$ is globally conserved at a given order in $1/\mu$. Since $\rho_c^{(0)}$ is a constant in space time we expect from Eqs. (56) and (58) that $v_c^{(0)}$ evolves according to Eq. (57) to ensure that the current density is conserved. As a result, there is a charge motion associated with the momentum but in the large λ limit this "recoil" momentum is absorbed by the whole charge lattice.

C. $O(1/\mu)$

For the sake of brevity we do not write down the equations at this order but make some comments instead. In the previous order, $O(1)$ we noticed [see Eqs. (56) and (58)] that spin degrees of freedom evolve as the charge is essentially frozen and at that order there is no feedback of the charge degrees of freedom on the spin. However, in the order $O(1/\mu)$ we have feedback terms in both evolution equations for ρ_s and v_s. As an example we have $\dot{\rho}_s^{(1)} = -\partial_x\{\ldots + 2v_s^{(0)}\rho_c^{(1)} + v_c^{(0)}\rho_s^{(0)} + \ldots\}$ and $\dot{v}_s^{(1)} = -\partial_x\{\ldots + v_c^{(0)}v_s^{(0)} + \frac{\pi^2}{2}\rho_c^{(1)}\rho_s^{(0)} + \ldots.\}$ which clearly show that there is a charge feedback into the spin sector.

D. Evolution equations for Haldane-Shastry model from the freezing trick

The shortest way to evolution equations for Haldane-Shastry model is to take $\lambda \to \infty$ limit directly in Riemann-Hopf Eq. (43). After rescaling time $t = \tau/\mu$ we have

$$\tilde{k}_\tau + \tilde{k}\tilde{k}_x = 0, \tag{59}$$

where $\tilde{k} = k/\mu = k/(\lambda + 1/2)$. In the large λ limit we have using Eqs. (48) and (49) $\tilde{k}_{R\uparrow,L\uparrow} \to \pm\pi\rho_c$ and $\tilde{k}_{R\downarrow,L\downarrow} = -2v_s \pm 2\pi\rho_\downarrow$. Then the Eq. (59) gives evolution equations for Haldane-Shastry model with Eqs. (D12) and (D15).

VII. ILLUSTRATIONS

It is relatively simple to obtain the evolution of arbitrary (smooth) initial density and velocity profiles solving equations of the gradientless hydrodynamics [Eq. (39)] numerically. One can do it very effectively using the fact that the dynamics is separated into four Riemann-Hopf Eq. (43) and using their general solutions (45). In this section we give numerical results for charge and spin dynamics corresponding to a relaxation of a (spin) polarized center. These results show that due to nonlinearity of equations spin can drag charge in spin-Calogero model. We notice here that in the examples considered in this section the dynamics belongs to CO regime.[38]

A. Charge dynamics in a spin-singlet sector

As a first example we consider initial conditions $\rho_s, v_s = 0$ and some arbitrary initial conditions for ρ_c and v_c. It is easy to see from Eq. (39) that spin density and spin velocity remain zero at any time while charge degrees of freedom satisfy

$$\dot{\rho}_c = -\partial_x(\rho_c v_c),$$

$$\dot{v}_c = -\partial_x\left\{\frac{v_c^2}{2} + \frac{\pi^2\left(\lambda + \frac{1}{2}\right)^2\rho_c^2}{2}\right\}. \tag{60}$$

Hydrodynamics [Eq. (60)] is identical to the one of the Calogero-Sutherland model with one species of particles (except for the change $\lambda + 1 \to \lambda + 1/2$). It can be written as a system of two Riemann-Hopf Eq. (43) for fields $v_c \pm \pi(\lambda + 1/2)\rho_c$.

We conclude that the charge dynamics does not affect spin in a spin-singlet state at least in the gradientless limit. It is interesting to see how spin dynamics affects the charge one.

B. Dynamics of a polarized center

To see how spin drags charge we start with an initial configuration with static and uniform charge background. We assume that initially there is no spin current but there is a nonzero polarization given by a Lorentzian profile,

$$t = 0: \rho_c = 1, \quad v_c = 0, \quad v_s = 0, \quad \rho_s = \frac{h}{1 + (x/a)^2}, \tag{61}$$

i.e., there is an excess of particles with spin up over particles with spin down near the origin. The maximal polarization is h and a half width of the polarized center is a. As an illus-

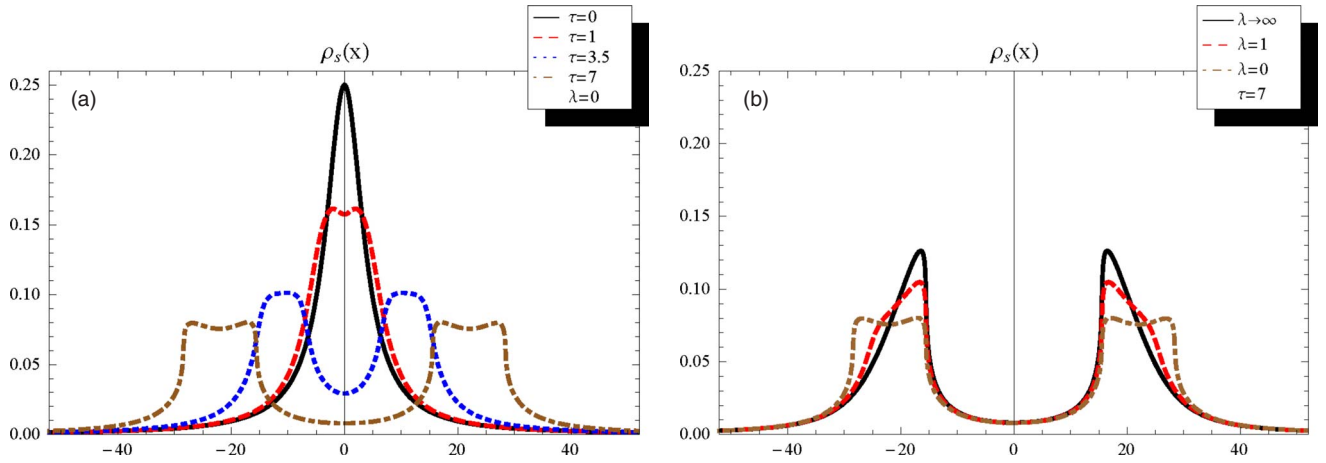

FIG. 5. (Color online) *Left panel*: Spin dynamics of polarized center for free fermions. The initial charge-density profile is constant and the initial spin-density profile is a Lorentzian (61) of a height $h=0.25$ and a half width $a=4$. Profiles at times $\tau=t/2=0,1,3.5,7$ are shown. *Right panel*: A snapshot of spin density at time $t=\tau/(\lambda+1/2)$ for $\tau=7$ for $\lambda=0,1,\infty$.

tration of spin and charge dynamics we present a solution of Eq. (39) with initial conditions [Eq. (61)] corresponding to $h=0.25$ and $a=4$. Some important comments are in order.

1. Applicability of gradientless hydrodynamics

The hydrodynamic equations we use [Eq. (39)] neglect gradient corrections and, therefore, are approximate. They can be applied only under the condition that the neglected higher gradient terms are small compared to the terms taken into account in Eq. (39). Of course, the exact criteria can be written only when the form of the higher gradient terms are known explicitly. Here, we are going to use a much simpler criterion. We require that all fields change slowly at the scale of the interparticle spacing. The uniform background $\rho_c=1$ defines the interparticle spacing and the characteristic scale for hydrodynamic fields to be 1 and we require $\partial_x f \ll 1$ for all fields at all x and t that we consider.

One can easily check that $\partial_x \rho_s(x,t=0) \ll 0.1$ for all x with the initial profile [Eq. (61)] (in fact, the maximal derivative is approximately 0.041). Because of the gradient catastrophe this condition will be broken at some time and we can trust the results obtained from Eq. (39) only up to that time. To be well within this criterion all our fields satisfy $\partial_x f < 0.3$ at any given time.

Let us start with the solutions for the case of free fermions, i.e., $\lambda=0$.

2. Free fermions with spin: $\lambda=0$

We present the results for spin and charge dynamics of free fermions with polarized center initial conditions [Eq. (61)] on left panels of Figs. 5 and 6. The profiles $\rho_s(x)$ and $\rho_c(x)-1$ are shown as functions of x for times $\tau=0,1,3,5,7$, respectively. Here we use a rescaled time $\tau=(\lambda+1/2)t=t/2$ for future convenience.

The dynamics is separated into four Riemann-Hopf equations for each Fermi momenta. The initial conditions [Eq. (61)] can be written as Lorentzian peaks for each of the four Fermi momenta of fermions and all four Fermi velocities are

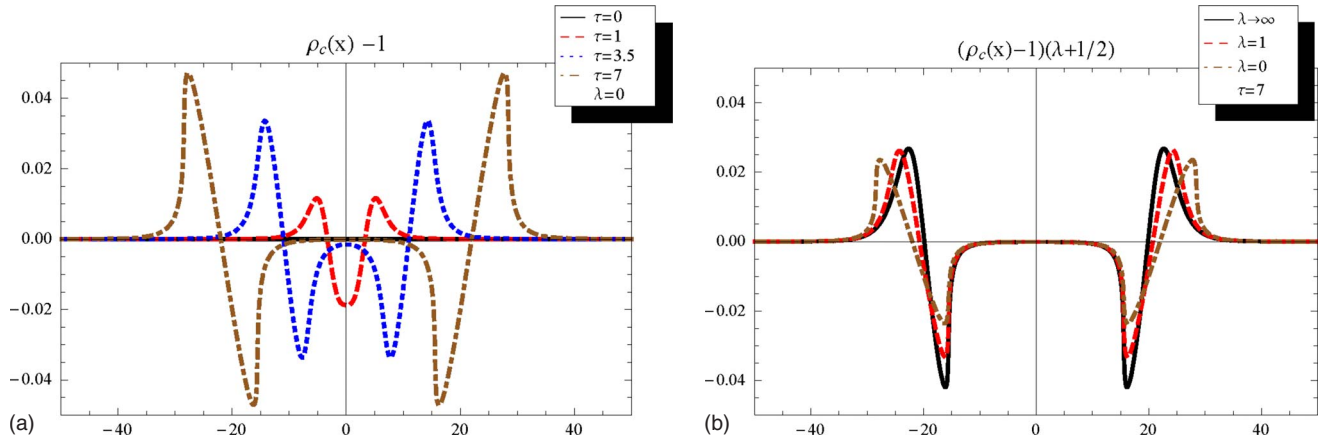

FIG. 6. (Color online) *Left panel*: Charge dynamics of polarized center for free fermions. The initial charge-density profile is constant and the initial spin-density profile is a Lorentzian (61) of a height $h=0.25$ and a half width $a=4$. Profiles at times $\tau=t/2=0,1,3.5,7$ are shown. *Right panel*: A snapshot of a rescaled charge density $(\lambda+1/2)(\rho_c-1)$ at time $t=\tau/(\lambda+1/2)$ for $\tau=7$ for $\lambda=0,1,\infty$.

different. This results in a splitting of an initial Lorentzian peak into four peaks at larger times which can be easily seen on the left panel of Fig. 5. In addition to this linear effects the nonlinear effects of steepening the wave front can also be seen. The latter will render gradientless hydrodynamics inapplicable at later times.

The drag of charge by spin clearly seen in Fig. 6 has an essentially nonlinear nature. There is an excess (deficit) of particles with spin up (down) at the origin at the initial moment. The particles with spin up will move away from the center while spin-down particles will move toward the center. However, the average velocity of spin-up particles is larger than the average velocity of spin-down particles as it is proportional to the density of those particles. Therefore, the initial motion of particles away and toward the origin creates a charge depletion in the center and charge-density maxima away from that depletion. This gives a qualitative explanation of the picture of charge dragged by spin which is shown in the left panel of Fig. 6. Notice that in this explanation we used the dependence of propagation velocity on the amplitude of the wave—an essentially nonlinear effect.

3. λ dependence of spin and charge dynamics

To see the effects of the interaction on spin and charge dynamics we show the spin- and charge-density profiles at a fixed time for different values of the coupling constant λ in the right panels of Figs. 5 and 6, respectively. It is convenient to use the scaling dictated by the $\lambda \to \infty$ limit considered in detail in Sec. VI. Namely, we use a rescaled time $\tau=(\lambda+1/2)t$ and rescale the deviation of the charge density from the uniform by background plotting $(\lambda+1/2)(\rho_c-1)$ for the charge density. The charge and density profiles found at $\tau=7$ are remarkably close for λ ranging from the free fermion case $\lambda=0$ to the limit of Haldane-Shastry model $\lambda \to \infty$.

The results confirm that the effect of spin dynamics on charge is suppressed by $1/\lambda$ for large λ. For a given initial spin-density profile the maximal amplitude of charge deviation is of the order $1/(\lambda+1/2)$.

VIII. CONCLUSIONS

In this paper we considered a classical two-fluid hydrodynamics derived as a semiclassical limit of the quantum sCM defined in Eq. (17). The model [Eq. (17)] is essentially quantum as it involves identical particles and a particle permutation operator. There is an essential ambiguity in how one takes a semiclassical limit. Here we considered a limit which is obtained when the density of particles goes to infinity so that $\hbar\rho$ is kept finite in the limit $\hbar \to 0$. We have also neglected gradient corrections to hydrodynamic equations assuming that fields change very slowly on the scale of the interparticle spacing. With all these assumptions, hydrodynamic equations are obtained from the Bethe-ansatz solution of sCM. They have the simplest form when written in terms of fields corresponding to dressed Fermi momenta of Bethe ansatz. In terms of these fields [Eqs. (48) and (49)], equations separate into four independent Riemann-Hopf Eq. (43) which are trivially integrable.

We presented some particular solutions of the hydrodynamic equations illustrating interactions between spin and charge. There is no true spin-charge separation in sCM. However, in the limit of large coupling constant $\lambda \to \infty$ the spin degrees of freedom do not affect the dynamics of charge degrees of freedom. The spin dynamics then is described by the hydrodynamics of Haldane-Shastry spin model. We considered explicitly both this limit $(\lambda \to \infty)$ and the limit $(\lambda=0)$ of free fermions with spin.

The quantum-scattering phase of particles interacting via $1/x^2$ potential is momentum independent. Moreover, it is the same for particles of the same species and for particles of different species because of the SU(2) invariance of Eq. (17). It is well known that this allows one to describe sCM as a model of free exclusions—particles obeying an exclusion statistics.[7–10] We do not keep the SU(2) invariance of the original quantum model [Eq. (17)] explicitly when taking the classical limit. However, this invariance is responsible for the variable separation that we observed in our hydrodynamics. We note here that sCM can be generalized to the "multispecies Calogero model."[39] Because of the absence of the SU(2) invariance for a more general two-species Calogero model one does not have the separation of variables for the corresponding hydrodynamics.

The classical gradientless hydrodynamics derived in this paper captures a lot of the features of sCM. It is straightforward to generalize our results to the case of the SU(n) Calogero model and to use the gradientless hydrodynamic equations for problems where field gradients can be neglected. In a separate publication (see Ref. 40) we use these equations in instanton calculations for the computation of emptiness formation probability similar to what was done in Refs. 31 and 41.

However, some important features of the hydrodynamic description do require an account of gradient corrections. First of all, the exact hydrodynamic equations are expected not to have an exact separation of variables. The obtained Riemann-Hopf Eq. (43) acquire gradient corrections and four such equations written for Eqs. (48) and (49) are expected to be coupled by those gradient corrections similarly to the case of the one-species Calogero-Sutherland model.[42] Similarly, we expect that the equations with gradient corrections will have soliton solutions corresponding to quasiparticle excitations of the quantum model [Eq. (17)].[6,15,42]

The hydrodynamic description of quantum sCM has been addressed in Refs. 17 and 18 using the collective-field-theory approach. The comparison of our results with results of those works is not straightforward. One should apply the collective formulation of Refs. 17 and 18 to the states from an appropriate sector of coherent states and take a corresponding classical limit. It would be especially interesting to see how the three hydrodynamic regimes discussed here appear from Refs. 17 and 18. One can also recognize a lot of similar looking terms in quantum hydrodynamics of Refs. 18 and in our classical gradientless hydrodynamics. It would also be very important to understand the role of the degeneracy due to the Yangian symmetry in sCM on its hydrodynamics. The latter degeneracy was neglected in the classical hydrodynamics of this paper.

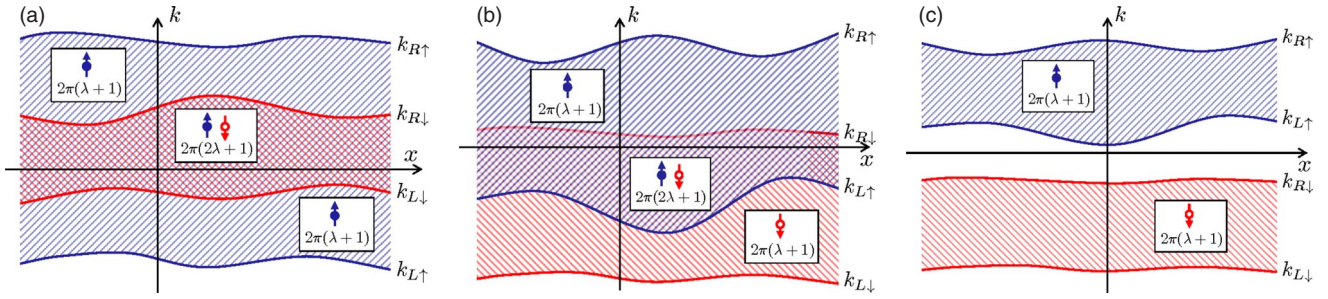

FIG. 7. (Color online) Phase-space diagrams of a hydrodynamic states characterized by four space-dependent Fermi momenta in three regimes CO, PO, and NO, respectively.

ACKNOWLEDGMENTS

We all have benefited from discussions with A. Polychronakos. A.G.A. is grateful to Y. Kato for a discussion of spin-charge (non)separation in spin-Calogero model. M.K. thanks D. Schneble for discussions of spin-charge dynamics in interacting systems. A.G.A. and F.F. would like to acknowledge the hospitality of the Galileo Galilei Center for Theoretical Physics (Florence, Italy) during the workshop on *Low-dimensional Quantum Field Theories and Applications*, where some interesting discussions took place. The work of A.G.A. was supported by the NSF under the Grant No. DMR-0348358.

APPENDIX A: ASYMPTOTIC BETHE-ANSATZ SOLUTION OF SPIN-CALOGERO MODEL AND SEPARATION OF VARIABLES IN HYDRODYNAMICS

The spin-Calogero model is solvable by asymptotic Bethe Ansatz (ABA).[6,28,43] This solution turns out to be the most convenient for our purposes.

The most important ingredient of ABA is the scattering phase which is given by

$$\theta(k) = \pi\lambda\ \mathrm{sgn}(k) \tag{A1}$$

for sCM. Here k is the relative momentum of two particles and the scattering phase does not depend on the species of particles. The expression for the dressed (true physical) momentum of the particle is given by

$$k(\kappa) = \frac{2\pi}{L}\left[\kappa + \frac{\lambda}{2}\int_{-\infty}^{\infty} \mathrm{sgn}(\kappa-\kappa')\nu(\kappa')\mathrm{d}\kappa'\right], \tag{A2}$$

where κ is an integer-valued noninteracting momentum of the particle (quantum number) and $\nu(\kappa)$ is the number of particles with quantum number κ [see Eq. (18)]. Here we replaced in the scattering phase $\mathrm{sgn}(k-k')$ by $\mathrm{sgn}(\kappa-\kappa')$.[44]

We immediately obtain from Eq. (A2)

$$L\frac{dk}{2\pi} = [1+\lambda\nu(\kappa)]d\kappa \tag{A3}$$

and

$$\nu_{\uparrow(\downarrow)}(\kappa)d\kappa = L\frac{\nu_{\uparrow(\downarrow)}(k)}{1+\lambda\nu(k)}\frac{dk}{2\pi}. \tag{A4}$$

We can see that the picture corresponding to Eq. (A4) in a single-particle phase space requires that the number of particles in the phase-space volume is given by $\frac{dxdk}{2\pi(\lambda+1)}$ if only one species is present and $\frac{dxdk}{2\pi(\lambda+1/2)}$ when both species are present. This justifies the picture we used (see Figs. 4 and 7).

It is easy to write down the expressions for the conserved quantities using Eq. (A4),

$$N_{\uparrow(\downarrow)} = L\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{\nu_{\uparrow(\downarrow)}(k)}{1+\lambda\nu(k)}, \tag{A5}$$

$$P = L\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{\nu(k)}{1+\lambda\nu(k)}k, \tag{A6}$$

$$P_s = L\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{\nu_s(k)}{1+\lambda\nu(k)}k, \tag{A7}$$

$$E = L\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{\nu(k)}{1+\lambda\nu(k)}\frac{k^2}{2}. \tag{A8}$$

Here P_s is a conserved quantity proportional to $\mathcal{L}_1^z$ introduced in Ref. 45

$$\hat{P}_s \equiv -i\sum_{j=1}^{N}\sigma_j^z\frac{\partial}{\partial x_j} - i\frac{\lambda}{2}\frac{\pi}{L}\sum_{j\neq l}\cot\frac{\pi}{L}(x_j-x_l)[\sigma_j^z-\sigma_l^z]P_{jl}. \tag{A9}$$

One can think of, e.g., $P_\uparrow = (P+P_s)/2$ as of a sum of asymptotic values of momenta of spin-up particles. We have replaced summations by integrations as we need only continuous versions of these formulas. It can be shown that Eqs. (A5), (A6), and (8) are equivalent to Eqs. (22), (19), and (20) with the relation between physical and noninteracting momenta given by Eq. (A2). Moreover, because the measure of integration $\frac{dk}{2\pi}\frac{\nu_{\uparrow(\downarrow)}(k)}{1+\lambda\nu(k)}$ is a piecewise constant for the two-step distribution [Eq. (24)], one naturally obtains integrals of motion in a form which is completely separated in terms of Fermi momenta. Indeed for a two-step distribution [Eq. (24)]

$$\nu_\alpha = \begin{cases} 1, & \text{if } k_{L\alpha} < k < k_{R\alpha} \\ 0, & \text{otherwise} \end{cases}, \tag{A10}$$

where $\alpha = \uparrow, \downarrow$. In the CO regime [Eq. (50)] we have

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\nu_\uparrow(k)}{1+\lambda\nu(k)} f(k) = \int_{k_{L\uparrow}}^{k_{L\downarrow}} \frac{dk}{2\pi} \frac{1}{1+\lambda} f(k) + \int_{k_{L\downarrow}}^{k_{R\downarrow}} \frac{dk}{2\pi} \frac{1}{1+2\lambda} f(k) \times + \int_{k_{R\downarrow}}^{k_{R\uparrow}} \frac{dk}{2\pi} \frac{1}{1+\lambda} f(k),$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\nu_\downarrow(k)}{1+\lambda\nu(k)} f(k) = \int_{k_{L\downarrow}}^{k_{R\downarrow}} \frac{dk}{2\pi} \frac{1}{1+2\lambda} f(k), \quad \text{(A11)}$$

where $f(k)$ is an arbitrary function. In particular, we obtain for the densities

$$2\pi(\lambda+1)\frac{N}{L} = k_{R\uparrow} - k_{L\uparrow} + \frac{1}{2\lambda+1}(k_{R\downarrow} - k_{L\downarrow}), \quad \text{(A12)}$$

$$2\pi(\lambda+1)\frac{N_s}{L} = k_{R\uparrow} - k_{L\uparrow} - (k_{R\downarrow} - k_{L\downarrow}), \quad \text{(A13)}$$

$$4\pi(\lambda+1)\frac{P}{L} = k_{R\uparrow}^2 - k_{L\uparrow}^2 + \frac{1}{2\lambda+1}(k_{R\downarrow}^2 - k_{L\downarrow}^2), \quad \text{(A14)}$$

$$4\pi(\lambda+1)\frac{P_s}{L} = k_{R\uparrow}^2 - k_{L\uparrow}^2 - (k_{R\downarrow}^2 - k_{L\downarrow}^2), \quad \text{(A15)}$$

$$12\pi(\lambda+1)\frac{E}{L} = k_{R\uparrow}^3 - k_{L\uparrow}^3 + \frac{1}{2\lambda+1}(k_{R\downarrow}^3 - k_{L\downarrow}^3). \quad \text{(A16)}$$

So far we presented the values of the conserved quantities for the sCM in terms of dressed Fermi momenta. They are given by linear combinations of Fermi momenta raised to the same power. There are infinitely many integrals of motion of this type and they are all in involution (commute with each other). The latter is a pretty stringent requirement and we assume that the only way to satisfy it is to require that the corresponding classical hydrodynamic fields have the following Poisson's brackets:

$$\{k_\alpha(x), k_\beta(y)\} = 2\pi s_\alpha \delta_{\alpha\beta} \delta'(x-y), \quad \text{(A17)}$$

where α runs over all Fermi points and s_α are some numbers to be determined. We can determine these numbers, e.g., in the following way. The density of current j (momentum per unit length) from Eq. (A14) can be defined by

$$j(x) = \frac{1}{4\pi(\lambda+1)}\left[k_{R\uparrow}^2 - k_{L\uparrow}^2 + \frac{1}{2\lambda+1}(k_{R\downarrow}^2 - k_{L\downarrow}^2)\right]. \quad \text{(A18)}$$

The total momentum of the system is a generator of the translation algebra $\{P, q(y)\} = \partial_y q(y)$, where $q(y)$ is any field. For the current density we should have

$$\{j(x), q(y)\} = q(x)\delta'(x-y). \quad \text{(A19)}$$

Taking $q(y)$ to be $k_\alpha(y)$ and combining Eq. (A19) with Eq. (A17) we fix the unknown coefficients s_α

$$s_{R\uparrow} = -s_{L\uparrow} = \lambda + 1,$$

$$s_{R\downarrow} = -s_{L\downarrow} = (\lambda+1)(2\lambda+1). \quad \text{(A20)}$$

Computing Poisson's bracket of the hydrodynamic Hamiltonian [obtained from Eq. (A16)]

$$H = \frac{1}{12\pi(\lambda+1)} \int dx \left[k_{R\uparrow}^3 - k_{L\uparrow}^3 + \frac{1}{2\lambda+1}(k_{R\downarrow}^3 - k_{L\downarrow}^3)\right] \quad \text{(A21)}$$

with $k_\alpha(x)$ we obtain Riemann-Hopf Eq. (43) for every Fermi momentum field $k_\alpha(x,t)$.

APPENDIX B: HYDRODYNAMIC VELOCITIES

In Appendix A we did not use the notion of hydrodynamic velocity. Instead, our hydrodynamic equations were written directly in terms of dressed Fermi momentum fields $k_\alpha(x,t)$. We also know how to express other quantities such as density, momentum, energy, etc., in terms of these variables. Let us now find the expressions for the velocity fields $v_{\uparrow,\downarrow}$. We focus on the CO regime here and consider other regimes in Appendix C.

First of all we, give the expressions for the conserved densities and conserved current densities which can be found from Eqs. (A12) and (A13) as

$$\rho_\uparrow = \frac{\rho+\rho_s}{2} = \frac{1}{2\pi(\lambda+1)}\left[k_{R\uparrow} - k_{L\uparrow} - \frac{\lambda}{2\lambda+1}(k_{R\downarrow} - k_{L\downarrow})\right],$$

$$\rho_\downarrow = \frac{\rho-\rho_s}{2} = \frac{1}{2\pi(2\lambda+1)}(k_{R\downarrow} - k_{L\downarrow}),$$

$$j_\uparrow = \frac{j+j_s}{2} = \frac{1}{4\pi(\lambda+1)}\left[k_{R\uparrow}^2 - k_{L\uparrow}^2 - \frac{\lambda}{2\lambda+1}(k_{R\downarrow}^2 - k_{L\downarrow}^2)\right], \quad \text{(B1)}$$

$$j_\downarrow = \frac{j-j_s}{2} = \frac{1}{2\pi(2\lambda+1)}(k_{R\downarrow}^2 - k_{L\downarrow}^2). \quad \text{(B2)}$$

In hydrodynamics, the velocities are defined as variables conjugated to the conserved momenta. Namely, the differential of the energy density defines chemical potentials and velocities as

$$d\epsilon = \mu_\uparrow d\rho_\uparrow + \mu_\downarrow d\rho_\downarrow + v_\uparrow^h dj_\uparrow + v_\downarrow^h dj_\downarrow. \quad \text{(B3)}$$

Using the energy density obtained from Eq. (A21) we have

$$d\epsilon = \frac{1}{4\pi(\lambda+1)}\left[k_{R\uparrow}^2 dk_{R\uparrow} - k_{L\uparrow}^2 dk_{L\uparrow} + \frac{1}{2\lambda+1}(k_{R\downarrow}^2 dk_{R\downarrow} - k_{L\downarrow}^2 dk_{L\downarrow})\right] \quad \text{(B4)}$$

and using Eqs. (B1) and (B2) one can determine $\mu_{\uparrow,\downarrow}$ and $v_{\uparrow,\downarrow}$. The hydrodynamic velocities are given by linear combinations of Fermi momenta[46]

$$v_{\uparrow}^{h} = \frac{1}{2}(k_{R\uparrow} + k_{L\uparrow}),$$

$$v_{\downarrow}^{h} = \frac{1}{2(\lambda+1)}[\lambda(k_{R\uparrow} + k_{L\uparrow}) + (k_{R\downarrow} + k_{L\downarrow})]. \tag{B5}$$

Using Eqs. (A17) and (A20) one can check that the velocities [Eq. (B5)] have canonical Poisson's brackets with densities [Eq. (B1)][46]

$$\{\rho_{\alpha}(x), v_{\beta}(y)\} = \delta_{\alpha\beta}\delta'(x-y), \tag{B6}$$

where $\alpha, \beta = \uparrow, \downarrow$. The other Poisson's brackets vanish.

The hydrodynamic velocities [Eq. (B5)] are precisely the ones used in the main body of this paper for CO regime $v_{\uparrow,\downarrow} = v^{h}_{\uparrow,\downarrow}$. Equations (48) and (49) are the inverse to Eqs. (B1) and (B5). Interestingly, in the CO regime the velocities and densities of different species can be naturally (simply) written in terms of bare noninteracting momenta [Eqs. (29) and (30)].

The current density in terms of densities and velocities follows from Eq. (A18) [compare with Eq. (31)]

$$j(x) = \rho_{\uparrow} v_{\uparrow} + \rho_{\downarrow} v_{\downarrow}. \tag{B7}$$

The density of "spin current" which follows from Eq. (A15) has a "correction" proportional to λ compared to the case of free fermions

$$j_s(x) = \rho_{\uparrow} v_{\uparrow} - \rho_{\downarrow} v_{\downarrow} + 2\lambda\rho_{\downarrow}(v_{\uparrow} - v_{\downarrow}). \tag{B8}$$

In this appendix we focused on CO regime. Of course, the formalism reviewed here is applicable to all three hydrodynamic regimes (CO, PO, and NO). We collect appropriate results in Appendix C.

APPENDIX C: HYDRODYNAMIC REGIMES FOR SPIN-CALOGERO MODEL

Depending on the relative order of four quantum numbers $\kappa_{R,L;\uparrow,\downarrow}$ we distinguish six different hydrodynamic regimes of the sCM. These regimes can be reduced to three essentially different ones exchanging $\uparrow \leftrightarrow \downarrow$. In this appendix we consider these three regimes and then combine all six cases.

Before we proceed, let us remark that the function $k(\kappa)$ defined in Eq. (A2) is monotonic and the order of the quantum numbers $\kappa_{R,L;\uparrow,\downarrow}$ is the same as the one of the physical dressed momenta $k_{R,\uparrow} = k(\kappa_{R,\uparrow})$, etc. Therefore, we can use the latter to define hydrodynamic regimes instead of the bare momenta κ.

1. Conserved densities and dressed Fermi momenta

Let us consider generally some integrable system which has two infinite families of mutually commuting conserved quantities. We assume further that the densities of these quantities are given in terms of four dressed Fermi momenta $k_{\alpha}(x)$ with $\alpha = 1,2,3,4$ as

$$j_n(x) = \frac{1}{n}\sum_{\alpha=1}^{4} a_{\alpha}[k_{\alpha}(x)]^n,$$

$$j_n^s(x) = \frac{1}{n}\sum_{\alpha=1}^{4} a_{\alpha} b_{\alpha}[k_{\alpha}(x)]^n. \tag{C1}$$

Here $n = 1,2,3,\ldots$ and a_{α}, b_{α} are constant coefficients. We assumed that the conserved densities can be expressed locally in terms of k_{α} and neglected gradient corrections.

We identify the first several integrals with densities, currents, and the energy as

$$j_1(x) = \rho(x),$$

$$j_1^s(x) = \rho_s(x),$$

$$j_2(x) = j(x),$$

$$j_2^s(x) = j_s(x),$$

$$j_3(x) = 2\epsilon(x). \tag{C2}$$

We notice here that due to Eqs. (A5)–(A8) the identifications [Eq. (C2)] [with Eq. (C1)] are valid for sCM model in all its regimes. The higher order conserved densities [Eq. (C1)] correspond to conserved quantities of sCM introduced in Ref. 45.

The requirement of vanishing Poisson's brackets between conserved quantities is very restrictive. It can be resolved by requiring canonical Poisson's brackets between Fermi momenta [Eq. (A17)]. If Eq. (A17) is valid, it is easy to check that $\{\int dx j_n(x), \int dy j_m^s(y)\} = 0$, etc. Using the fact that the total current is the generator of translations [Eq. (A19)] we can fix the coefficients s_{α} in Eq. (A17) as $2\pi s_{\alpha} = 1/a_{\alpha}$ and obtain

$$\{k_{\alpha}(x), k_{\beta}(y)\} = \frac{1}{a_{\alpha}}\delta_{\alpha\beta}\delta'(x-y). \tag{C3}$$

Using the Poisson's brackets [Eq. (C3)] and the Hamiltonian $H = \int dx \epsilon(x)$ with Eqs. (C2) and (C1) it is easy to obtain the Riemann-Hopf evolution equations for the dressed Fermi momenta

$$\partial_t k_{\alpha} + k_{\alpha}\partial_x k_{\alpha} = 0, \quad \text{for} \quad \alpha = 1,2,3,4 \tag{C4}$$

and the evolution equations for all conserved densities as

$$\partial_t j_n + \partial_x j_{n+1} = 0,$$

$$\partial_t j_n^s + \partial_x j_{n+1}^s = 0. \tag{C5}$$

In the hydrodynamic regime only four of the densities are algebraically independent (as there are only four dressed Fermi momenta). Therefore, one can find constitutive relations, i.e., express the energy density in terms of ρ, ρ_s, j, and j_s. Alternatively, one can use hydrodynamic velocities v^h and v_s^h defined by Eq. (B3) instead of currents j, j_s.

We can see that the hydrodynamics [Eqs. (C1)–(C3)] is fully defined by coefficients a_{α}, b_{α}. In fact, these coefficients are not totally independent. Requiring that densities ρ and ρ_s

TABLE I. Summary of three regimes.

	α	$L\uparrow$	$R\uparrow$	$L\downarrow$	$R\downarrow$	
CO	$2\pi(\lambda+1)a_\alpha$	-1	1	$-\frac{1}{2\lambda+1}$	$\frac{1}{2\lambda+1}$	$k_{L\uparrow}<k_{L\downarrow}<k_{R\downarrow}<k_{R\uparrow}$
	b_α	1	1	$-(2\lambda+1)$	$2\lambda+1$	
PO	$2\pi(\lambda+1)a_\alpha$	-1	$\frac{1}{2\lambda+1}$	$-\frac{1}{2\lambda+1}$	1	$k_{L\downarrow}<k_{L\uparrow}<k_{R\downarrow}<k_{R\uparrow}$
	b_α	1	$2\lambda+1$	$-(2\lambda+1)$	-1	
NO	$2\pi(\lambda+1)a_\alpha$	-1	1	-1	1	$k_{L\downarrow}<k_{R\downarrow}<k_{L\uparrow}<k_{R\uparrow}$
	b_α	1	1	-1	-1	

have vanishing Poisson's brackets with themselves and with each other gives three relations between the coefficients

$$\sum_\alpha a_\alpha = 0,$$

$$\sum_\alpha b_\alpha a_\alpha = 0,$$

$$\sum_\alpha b_\alpha^2 a_\alpha = 0. \tag{C6}$$

For CO, PO, and NO regimes of sCM these coefficients are summarized in the Table I. These coefficients do satisfy relations (C6).

The matrix of Poisson's brackets of the dressed Fermi momenta k_α [Eq. (C3)] is diagonal but not proportional to the unit matrix. It is interesting that the Poisson's brackets of bare momenta κ_α satisfy

$$\{\kappa_\alpha(x),\kappa_\beta(y)\} = (-1)^\alpha \frac{L^2}{2\pi}\delta_{\alpha\beta}\delta'(x-y). \tag{C7}$$

One then obtains that the velocities introduced in Eqs. (29) and (30) are canonically conjugate to the corresponding densities and can be written as linear combinations of κ_α (and of k_α). The velocities [Eqs. (29) and (30)] are defined just as conjugate variables to the densities. This definition is not unique. One can always shift $v_\uparrow \to v_\uparrow + 2\pi\gamma\rho_\downarrow$ and $v_\downarrow \to v_\downarrow - 2\pi\gamma\rho_\uparrow$ with any number γ without changing Poisson's brackets. The particular choice of variables [Eqs. (29) and (30)] is convenient because it defines velocities continuously across all hydrodynamic regimes. Moreover, we have

$$v_{\uparrow,\downarrow} = v^h_{\uparrow,\downarrow}, \quad \text{for CO},$$

$$v_{\uparrow,\downarrow} = v^h_{\uparrow,\downarrow} \pm \pi\lambda\rho_{\downarrow,\uparrow}, \quad \text{for NO}. \tag{C8}$$

In PO regime the hydrodynamic velocities are not linear combinations of k_α and their relations to the conjugated variables $v_{\uparrow,\downarrow}$ used in this paper are more complicated.

2. Complete overlap regime (CO)

The complete overlap regime corresponds to the case when

$$-\frac{\pi}{2}|\rho_s| < v_s < \frac{\pi}{2}|\rho_s|. \tag{C9}$$

In this case the support of $\nu_\downarrow$ is a subset of the support of $\nu_\uparrow$ (or vice versa). In the main body of the paper we mostly concentrated on this case but for convenience we recap the main formulas in this appendix as well. The dressed momenta [Eq. (A2)] in the CO regime for $\rho_s > 0$, i.e., for the ordering

$$k_{L\uparrow} < k_{L\downarrow} < k_{R\downarrow} < k_{R\uparrow} \tag{C10}$$

are

$$\begin{aligned} k_{R\uparrow,L\uparrow} &= v_\uparrow \pm \pi[(\lambda+1)\rho_\uparrow + \lambda\rho_\downarrow] \\ &= v_\uparrow \pm \pi\rho_\uparrow \pm \lambda\pi\rho_c, \end{aligned}$$

$$\begin{aligned} k_{R\downarrow,L\downarrow} &= (\lambda+1)v_\downarrow - \lambda v_\uparrow \pm \pi(2\lambda+1)\rho_\downarrow \\ &= v_\downarrow \pm \pi\rho_\downarrow + \lambda(-2v_s \pm 2\pi\rho_\downarrow). \end{aligned} \tag{C11}$$

Poisson's brackets of k_α are given by Eq. (C3) with coefficients from the Table I. One can express all conserved densities [Eq. (C1)] in terms of dressed Fermi momenta using the Table I. For example, the Hamiltonian [see Eq. (C2)] reads

$$H_{\mathrm{CO}} = \frac{1}{12\pi(\lambda+1)}\int dx\left[k_{R\uparrow}^3 - k_{L\uparrow}^3 + \frac{1}{2\lambda+1}(k_{R\downarrow}^3 - k_{L\downarrow}^3)\right], \tag{C12}$$

$$\begin{aligned} &= \int dx\left\{\frac{1}{2}\rho_\uparrow v_\uparrow^2 + \frac{1}{2}\rho_\downarrow v_\downarrow^2 + \frac{\lambda}{2}\rho_\downarrow(v_\uparrow - v_\downarrow)^2 + \frac{\pi^2\lambda^2}{6}\rho_c^3\right. \\ &\left. + \frac{\pi^2}{6}(\rho_\uparrow^3 + \rho_\downarrow^3) + \frac{\lambda\pi^2}{6}(2\rho_\uparrow^3 + 3\rho_\uparrow^2\rho_\downarrow + 3\rho_\downarrow^3)\right\}. \end{aligned} \tag{C13}$$

The evolution equations are given by Eq. (C4) and can also be recast in terms of equations for densities and velocities [Eqs. (38) and (39)].

3. Partial overlap regime (PO)

There are two regimes when the supports of $\nu_\uparrow$ and $\nu_\downarrow$ only partially overlap. Here we concentrate on the case for which

TABLE II. Classification of different regimes: + indicates that the field takes positive values and − indicates that it is negative. A blank means that its sign is arbitrary.

k inequality	v_s	ρ_s	$\xi_1=v_s+\frac{\pi}{2}\rho_s$	$\xi_2=v_s-\frac{\pi}{2}\rho_s$	$\chi_1=v_s+\frac{\pi}{2}\rho_c$	$\chi_2=v_s-\frac{\pi}{2}\rho_c$	v_s inequality	Regime
$k_{L\uparrow}<k_{R\uparrow}<k_{L\downarrow}<k_{R\downarrow}$	−		−	−	−	−	$v_s<-\frac{\pi}{2}\rho_c$	NO
$k_{L\uparrow}<k_{L\downarrow}<k_{R\uparrow}<k_{R\downarrow}$	−		−	−	+	−	$-\frac{\pi}{2}\rho_c<v_s<-\frac{\pi}{2}\lvert\rho_s\rvert$	PO
$k_{L\downarrow}<k_{L\uparrow}<k_{R\uparrow}<k_{R\downarrow}$		−	−	+	+	−	$\frac{\pi}{2}\rho_s<v_s<-\frac{\pi}{2}\rho_s$	CO
$k_{L\uparrow}<k_{L\downarrow}<k_{R\downarrow}<k_{R\uparrow}$		+	+	−	+	−	$-\frac{\pi}{2}\rho_s<v_s<\frac{\pi}{2}\rho_s$	CO
$k_{L\downarrow}<k_{L\uparrow}<k_{R\downarrow}<k_{R\uparrow}$	+		+	+	+	−	$\frac{\pi}{2}\lvert\rho_s\rvert<v_s<\frac{\pi}{2}\rho_c$	PO
$k_{L\downarrow}<k_{R\downarrow}<k_{L\uparrow}<k_{R\uparrow}$	+		+	+	+	+	$\frac{\pi}{2}\rho_c<v_s$	NO

$$\frac{\pi}{2}|\rho_s|<v_s<\frac{\pi}{2}\rho_c, \tag{C14}$$

corresponding to the ordering

$$k_{L\downarrow}<k_{L\uparrow}<k_{R\downarrow}<k_{R\uparrow}. \tag{C15}$$

The other PO regime can be obtained by exchanging up and down particles, i.e., by changing $v_s\to -v_s$. In this case the dressed momenta [Eq. (A2)] are

$$\begin{aligned}
k_{L\downarrow}&=v_\downarrow-\pi(\lambda+1)\rho_\downarrow-\pi\lambda\rho_\uparrow=v_\downarrow-\pi\rho_\downarrow-\lambda\pi\rho_c,\\
k_{L\uparrow}&=v_\uparrow+\lambda(v_\uparrow-v_\downarrow)-\pi(2\lambda+1)\rho_\uparrow\\
&=v_\uparrow-\pi\rho_\uparrow+\lambda(2v_s-2\pi\rho_\uparrow),\\
k_{R\downarrow}&=v_\downarrow-\lambda(v_\uparrow-v_\downarrow)+\pi(2\lambda+1)\rho_\downarrow\\
&=v_\downarrow+\pi\rho_\downarrow-\lambda(2v_s-2\pi\rho_\downarrow),\\
k_{R\uparrow}&=v_\uparrow+\pi(\lambda+1)\rho_\uparrow+\pi\lambda\rho_\downarrow=v_\uparrow+\pi\rho_\uparrow+\lambda\pi\rho_c
\end{aligned} \tag{C16}$$

and the Hamiltonian becomes [see Table I and Eqs. (C1) and (C2)]

$$H_{\mathrm{PO}}=\frac{1}{12\pi(\lambda+1)}\int dx\left[k_{R\downarrow}^3-k_{L\uparrow}^3+\frac{1}{2\lambda+1}(k_{R\uparrow}^3-k_{L\downarrow}^3)\right] \tag{C17}$$

$$\begin{aligned}
&=\int dx\left\{\frac{1}{2}\rho_\uparrow v_\uparrow^2+\frac{1}{2}\rho_\downarrow v_\downarrow^2+\lambda\pi\rho_\uparrow\rho_\downarrow(v_\downarrow-v_\uparrow)\right.\\
&-\frac{\lambda}{12\pi}[v_\uparrow-v_\downarrow-\pi(\rho_\uparrow+\rho_\downarrow)]^3+\frac{\pi^2\lambda^2}{6}(\rho_\uparrow+\rho_\downarrow)^3\\
&\left.+\frac{\pi^2}{6}(1+2\lambda)(\rho_\uparrow^3+\rho_\downarrow^3)\right\}.
\end{aligned} \tag{C18}$$

Poisson's brackets of k_α are given by Eq. (C3) with coefficients from the Table I and evolution equations are given by Eq. (C4).

4. No overlap regime (NO)

In this case, the supports of $\nu_\uparrow$ and $\nu_\downarrow$ do not overlap at all. For $v_s>0$ the ordering of dressed Fermi momenta is

$$k_{L\downarrow}<k_{R\downarrow}<k_{L\uparrow}<k_{R\uparrow} \tag{C19}$$

and momenta themselves are

$$\begin{aligned}
k_{R\uparrow,L\uparrow}&=v_\uparrow+\pi\lambda\rho_\downarrow\pm\pi(\lambda+1)\rho_\uparrow=v_\uparrow\pm\pi\rho_\uparrow\pm\lambda\pi\rho_{c,s},\\
k_{R\downarrow,L\downarrow}&=v_\downarrow-\pi\lambda\rho_\uparrow\pm\pi(\lambda+1)\rho_\downarrow=v_\downarrow\pm\pi\rho_\downarrow-\lambda\pi\rho_{s,c}
\end{aligned} \tag{C20}$$

and the Hamiltonian becomes [see Table I and Eqs. (C1) and (C2)]

$$H_{\mathrm{NO}}=\frac{1}{12\pi(\lambda+1)}\int dx[k_{R\uparrow}^3-k_{L\uparrow}^3+k_{R\downarrow}^3-k_{L\downarrow}^3], \tag{C21}$$

$$\begin{aligned}
&=\int dx\left\{\frac{1}{2}\rho_\uparrow v_\uparrow^2+\frac{1}{2}\rho_\downarrow v_\downarrow^2+\lambda\pi\rho_\uparrow\rho_\downarrow(v_\uparrow-v_\downarrow)\right.\\
&\left.+\frac{\pi^2}{6}(\lambda+1)^2(\rho_\uparrow+\rho_\downarrow)^3-\frac{\pi^2}{2}(1+2\lambda)\rho_\uparrow\rho_\downarrow(\rho_\uparrow+\rho_\downarrow)\right\}.
\end{aligned} \tag{C22}$$

Poisson's brackets of k_α are given by Eq. (C3) with coefficients from the Table I and evolution equations are given by Eq. (C4).

5. All cases combined

It is possible to combine all hydrodynamic regimes into relatively compact expressions introducing absolute values of hydrodynamic fields. A general Hamiltonian valid for all regimes takes a form

$$\begin{aligned}
H=\int dx&\left\{\frac{1}{2}\rho_\uparrow v_\uparrow^2+\frac{1}{2}\rho_\downarrow v_\downarrow^2+\frac{\pi^2}{6}(\rho_\uparrow^3+\rho_\downarrow^3)+\frac{\pi^2}{6}\lambda^2\rho_c^3\right.\\
&+\frac{\pi^2}{3}2\lambda(\rho_\uparrow^3+\rho_\downarrow^3)+\lambda\rho_c\xi_1\xi_2-\frac{\lambda}{3\pi}(|\xi_1|^3+|\xi_2|^3)\\
&\left.+\frac{\lambda}{3\pi}[|\chi_1|^3-\chi_1^3+|\chi_2|^3+\chi_2^3]\right\},
\end{aligned} \tag{C23}$$

where we introduced the following notations

$$\xi_{1,2}\equiv v_s\pm\frac{\pi}{2}\rho_s, \tag{C24}$$

$$\chi_{1,2} \equiv v_s \pm \frac{\pi}{2}\rho_c. \tag{C25}$$

Hamiltonian (C23) can be obtained from Eqs. (20) and (21) for the general case of a two-step distribution function $\nu_{\uparrow,\downarrow}(\kappa)$. We collect in the Table II the information necessary to go quickly from the general expression (C23) to the particular ones valid in separate regimes (CO, PO, or NO).

We can combine the evolution equations following from Eq. (C23) in the spin/charge basis [Eq. (12)] as

$$\dot{\rho}_c = -\partial_x\{\rho_c v_c + \rho_s v_s\}, \tag{C26}$$

$$\dot{\rho}_s = -\partial_x\left\{\rho_c v_s + \rho_s v_c - \frac{\lambda}{\pi}[\xi_1|\xi_1| + \xi_2|\xi_2| - \chi_1|\chi_1| - \chi_2|\chi_2|]\right\}, \tag{C27}$$

$$\dot{v}_c = -\partial_x\left\{\frac{v_c^2 + v_s^2}{2} + \frac{\pi^2}{8}[(4\lambda^2 + 2\lambda + 1)\rho_c^2 + (2\lambda + 1)\rho_s^2] + \frac{\lambda}{2}[\chi_1|\chi_1| - \chi_2|\chi_2|]\right\}, \tag{C28}$$

$$\dot{v}_s = -\partial_x\left\{v_c v_s + \frac{\pi^2}{4}(2\lambda + 1)\rho_c\rho_s - \frac{\lambda}{2}[\xi_1|\xi_1| - \xi_2|\xi_2|]\right\}. \tag{C29}$$

For CO and PO regimes Hamiltonian (C23) takes an especially simple form in terms of dressed momenta

$$H_{\text{CO and PO}} = \frac{1}{12\pi(2\lambda + 1)}\int dx\left\{k_{R\uparrow}^3 - k_{L\uparrow}^3 + k_{R\downarrow}^3 - k_{L\downarrow}^3 + \frac{\lambda}{(\lambda + 1)}[|k_{L\uparrow}^3 - k_{L\downarrow}^3| + |k_{R\uparrow}^3 - k_{R\downarrow}^3|]\right\}, \tag{C30}$$

which are related to density and velocity fields as

$$k_{R\uparrow,L\uparrow} = v_\uparrow \pm \pi(\lambda + 1)\rho_\uparrow + \lambda\chi_{1,2} \mp \lambda|\xi_{1,2}|,$$

$$k_{R\downarrow,L\downarrow} = v_\downarrow \pm \pi(1 + \lambda)\rho_\downarrow - \lambda\chi_{2,1} \mp \lambda|\xi_{1,2}|. \tag{C31}$$

As in the separate cases considered before, these momenta have canonical Poisson's brackets [Eq. (A17)] with

$$s_{R\uparrow,R\downarrow} = (\lambda + 1)[\lambda + 1 \pm \lambda\ \text{sgn}(\xi_1)],$$

$$s_{L\uparrow,L\downarrow} = -(\lambda + 1)[\lambda + 1 \mp \lambda\ \text{sgn}(\xi_2)] \tag{C32}$$

and evolve independently according to the Riemann-Hopf Eq. (C4).

APPENDIX D: HYDRODYNAMIC DESCRIPTION OF HALDANE-SHASTRY MODEL FROM ITS BETHE-ANSATZ SOLUTION

The HSM is a Heisenberg spin chain with long-ranged interaction defined by the Hamiltonian

$$H_{\text{HSM}} = \frac{1}{2}\sum_{j<l}\frac{\mathbf{K}_{jl}}{d(j-l)^2}, \tag{D1}$$

where K_{jl} is the spin-exchange operator,[47]

$$\mathbf{K}_{jl} = \frac{\vec{\sigma}_j \cdot \vec{\sigma}_l + 1}{2}, \tag{D2}$$

and $d(j) \equiv (N/\pi)|\sin(\pi j/N)|$ is the chord distance between two points on a lattice with N sites and periodic boundary conditions. The model [Eq. (D1)] has been introduced independently at the same time by Haldane[32] and by Shastry[33] and has been shown to be integrable. The energy spectrum of the HSM is equivalent to that of the Calogero-Sutherland model at $\lambda = 2$ but with a high degeneracy due to the Yangian symmetry.[48,49]

In this appendix we used the Bethe-Ansatz solution[32,50,51] to construct a gradientless hydrodynamic description for the HSM similarly to what we have done for the sCM model in Sec. IV and Appendix A. To this end, we consider a state with M overturned spins over an initial ferromagnetic configuration (say from up to down and $M < N/2$) and introduce M integer quantum numbers κ's to characterize the state in the Bethe-Ansatz formalism. As before such state can be described by a distribution function $\nu(\kappa) = 0, 1$, depending on whether that quantum number is present or not in the BA solution. Following Ref. 32 we impose a condition on the integer numbers: $|\kappa| < (N - M - 1)/2$.[52]

The scattering phase for the HSM is

$$\theta(k) = \pi\ \text{sgn}(k), \tag{D3}$$

which corresponds to setting $\lambda = 1$ into Eq. (A1).[53] Please note that since we are considering a lattice model, the momentum is defined within the Brillouin zone: $-\pi < k < \pi$, where we took the lattice spacing as unity.

At this point, all the derivations of Appendix A can be repeated step by step for the HSM just by setting everywhere $\lambda = 1$ and remembering that the momentum is always defined modulo 2π. In particular, the dressed momentum is

$$k(\kappa) = \frac{2\pi}{L}\left[\kappa + \frac{1}{2}\int \text{sgn}(\kappa - \kappa')\tilde{\nu}(\kappa')\mathrm{d}\kappa'\right], \tag{D4}$$

where again we replaced $\text{sgn}(k - k')$ by $\text{sgn}(\kappa - \kappa')$ (Ref. 44) and the distribution of the physical momenta is given by

$$\nu(\kappa)d\kappa = \frac{N}{4\pi}\nu(k)dk. \tag{D5}$$

In terms of this distribution function, the conserved quantities can be written as

$$M = N\int \frac{dk}{4\pi}\nu(k), \tag{D6}$$

$$P = N\int \frac{dk}{4\pi}\nu(k)k, \tag{D7}$$

$$E = E_0 + N\int \frac{dk}{4\pi}\nu(k)\frac{k^2}{2}, \tag{D8}$$

where the momentum is defined only modulo 2π. From now on, we will drop the constant energy shift E_0.

In a hydrodynamic description we assume a distribution of the uniform type

$$\nu(k) = \begin{cases} 1, & \text{if } -\pi < k_L < k < k_R < \pi, \\ 0, & \text{otherwise}, \end{cases} \tag{D9}$$

where $k_{R,L}$ are some numbers. Using Eq. (D9) and introducing space-dependent fields instead of constants we write Eqs. (D6) and (D7) as

$$M = \int dx \frac{k_R - k_L}{4\pi} = \int dx \rho, \tag{D10}$$

$$P = \int dx \frac{k_R^2 - k_L^2}{8\pi} = \int dx \rho v, \tag{D11}$$

which suggests the identification

$$k_{R,L} = v \pm 2\pi\rho. \tag{D12}$$

Then the hydrodynamic Hamiltonian follows from Eq. (D8),

$$H_{\mathrm{HSM}} = \int dx \frac{k_R^3 - k_L^3}{24\pi} = \int dx \left[\frac{1}{2}\rho v^2 + \frac{2}{3}\pi^2\rho^3 \right], \tag{D13}$$

which corresponds, as expected, to the (gradientless) hydrodynamic of a $\lambda = 1$ spinless Calogero-Sutherland model [Eq. (37)].

We think of slowly varying fields $\rho(x,t)$ and $v(x,t)$ as of classical fields obeying the Poisson relation $\{\rho(x), v(y)\} = \delta'(x-y)$. Then Eq. (D13) generates the evolutions equations

$$\dot{\rho} = -\partial_x(\rho v),$$

$$\dot{v} = -\partial_x\left(\frac{v^2}{2} + \frac{\pi^2}{2} 4\rho^2 \right). \tag{D14}$$

One can easily recognize in Eq. (D14) the hydrodynamics of spinless Calogero-Sutherland model [Eq. (37)] for $\lambda = 1$. The correspondence between eigenstates and eigenenergies of Haldane-Shastry model with $\lambda = 2$ spinless Calogero-Sutherland model has been noticed in the original paper.[32] The degeneracy of the states due to the SU(2) invariance and Yangian symmetry is lost in our classical hydrodynamics model.

For comparisons with the derivations from freezing trick[20] in Sec. VI we express Eqs. (D13) and (D17) in terms of ρ_s and v_s used in the main body of the paper. We identify the density $\rho = M/N = \rho_\downarrow$ as the density of spin-down particles and the velocity v as a velocity of spin-down particles relative to the static background of spin-up particles, i.e., $v = v_\uparrow - v_\downarrow = -2v_s$. The charge density corresponding to the lattice with spacing one is just $\rho_0 = 1$. We summarize

$$\rho = \rho_\downarrow = \frac{\rho_0 - \rho_s}{2}, \quad v = -2v_s, \quad \rho_0 = 1. \tag{D15}$$

Using Eq. (D15) we rewrite Eq. (D13) as

$$H_{\mathrm{HSM}} = \int dx \left\{ \rho_0 v_s^2 - \rho_s v_s^2 + \frac{\pi^2 \rho_0 \rho_s^2}{4} - \frac{\pi^2 \rho_s^3}{12} \right\}, \tag{D16}$$

where we neglected a constant and a term linear in ρ_s, which amounts to a shift in the chemical potential. The evolution equations for the spin density and spin velocity follow from Eqs. (D14) and (D15)

$$\dot{\rho}_s = -\partial_x\{2v_s\rho_0 - 2v_s\rho_s\},$$

$$\dot{v}_s = -\partial_x\left\{ -v_s^2 + \frac{\pi^2}{2}\rho_0\rho_s - \frac{\pi^2}{4}\rho_s^2 \right\}. \tag{D17}$$

We notice that the above Eqs. (D16) and (D17) is nothing but the strong-interaction limit of the sCM [Eqs. (53), (56), and (58)].

Finally, we remark that it is easy to check that the distribution function [Eq. (D9)] implies that $-\frac{\pi\rho_s}{2} \le v_s \le \frac{\pi\rho_s}{2}$ and therefore corresponds to the CO regime of spin-Calogero model.

Both in this appendix and in writing classical hydrodynamics for sCM we neglected the degeneracy of the corresponding quantum models due to the Yangian symmetry.[48,49] We assumed that during the evolution string states are not excited. Of course, the degeneracy plays a very important role for perturbed integrable systems and for the hydrodynamics at finite temperatures.

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[35] At least one will not be able to separate variables considering simple linear combinations of fields. To the best of our knowledge variables in sCM do not separate or, at least, an appropriate change in variables has not been found yet.
[36] We would like to thank A. Polychronakos who encouraged us to present this picture.
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